Eur. Phys. J. B **12**, 583–588 (1999)

THE EUROPEAN PHYSICAL JOURNAL B

EDP Sciences

Revealing of the simplest mechanisms of a structure formation

O.V. Kirillova[a]

Department of Theoretical Physics, St. Petersburg State University Ulyanovskaya str. 1, St.Petersburg, 198904 Russia

Received 10 March 1999 and Received in final form 24 May 1999

Abstract. In this paper, results of investigations of the simplest mechanisms of a structure formation are presented. In frameworks of the suggested model the main attention was focused on such characteristics as wiring of the system, clusters formation, dynamics of the wiring. The idea to take into account an influence of the environment factor is employed in the proposed model. Investigations of systems with such principle of a structure formation reveal that the system's dynamics has typical features of self-organized criticality phenomenon. In the avalanche-like processes, which occur in the wiring dynamics, a power law was found with the index close to 1.4. It is independent on the environment factor (which in a sense can be considered as system parameter). The system wiring is approximated pretty well by the Gaussian distribution. The size of the system does not play any role in the dynamics of the model.

PACS. 02.50.-r Probability theory, stochastic processes, and statistics – 05.90.+m Other topics in statistical physics, thermodynamics, and nonlinear dynamical systems

1 Introduction

Nowadays much attention is paid to studying and modeling of different systems and its dynamics [1]. Usually in such investigations the structure of a system is always supposed to be determined and constant, or totally random. On the other hand, there is a number of works devoted exactly to studying of structure formation [2] but, as a rule, it has very narrow specifics. In the sense of a system evolution, dynamics undoubtedly depends on the geometry of a system [3,4] and includes the structure formation as well. Moreover this property of the system dynamics have to be regarded as fundamental and general for a wide class of phenomena. So processes of a structure formation are very important in the world: from physics (atomic structure) to cosmology (structure of the Universe), from biology (evolution and morphogenesis) to social sciences (formation of different social groups).

The term *structure* can be understood in the sense of the spatial connection of objects' parts or interactions chains, like in biology: predator – prey relationship, foods chains, gene networks. In our opinion, the idea seems very attractive that the rise of a structure as a result of natural processes taking place within a system occurs in a way common for a wide class of phenomena.

As an example of the simplest model of a spatial structure formation, let us take the diffusion-limited aggregation model (DLA) [5]. As it was pointed out [5] such models demonstrate formation of the spatial structure with fractal properties. Another attempt to reveal a structure, had been done in the investigations [6] devoted to modeling of networks of gene expression. Much attention is paid to this question in graph theory [7]. But all the studies are only particular quests in a narrow field, and moreover today they seem to be absolutely disconnected.

From the standpoint of theoretical physics, it is out of the question that the topology plays a crucial role in that of what phenomena can take place in a model under study. So it would be interesting to construct a model in which a topology (a structure of interaction) arises in the course of dynamics. In this work we propose a simplest model describing structure formation in the dynamics of a stochastic system, and report results of the study. Let us emphasize that even simplest models reveal interesting regularities.

2 The model

Our world is undoubtedly very complex, so one might guess that to gain a detailed description it is necessary to write more and more sophisticated equations, to include more parameters and variables. On the one hand, it is true. But on the another, it is important to find the right point of view: to reveal fundamental mechanisms of a phenomenon and to check them out on a model. In the latter case, the simpler (but still nontrivial) model is, the more valuable it is since we can be sure that we gain a background for more detailed description. So our aim is not details but general comprehensive knowledge of how complex systems evolve. It is desirable that models would be free of fitting parameters so that one can study a mechanism itself but not parameter-dependent regimes. In our

[a] e-mail: kirill@heps.phys.spbu.ru

model, besides a size of a system we have only one parameter α that can be conditionally called *the factor of an environment influence*.

Dynamics of the model is following. At the beginning we have the system which consists of N nodes and zero links. On an iteration step a link between a pair of elements in the system appears with probability α and nothing happens with probability $1-\alpha$. Some time later we shall have a number of *clusters* (connected group of nodes) in the system. (The size of a cluster is the number of nodes, which forms the cluster.) At the K-th iteration step one of the nodes is randomly chosen. If it is not belong to any clusters, in other words has no links, there is possibility of a link appearing (with probability α) in the system and the situation is not changed with probability $1-\alpha$. If the chosen element belongs to a cluster, we "check" each link going from the element and with probability α it is preserved, while it is destroyed with probability $1-\alpha$. Thus at the K-th iteration step it is possible creation of one new link and destroying several links. And so on for the K+1, K+2, ... iteration steps.

Let us envisage the next analogy. It is possible to consider our model as a formation of a network of channels in a medium, which at the beginning is homogeneous (water in rock, electrical current in a medium with high resistance). By some means we increase pressure/potential in a point of the medium. If there are not channels going from the point then as the action result (if pressure/potential exceeded some threshold (it corresponds to parameter α of the model)) a hole occurs and the new channel is created. If a chosen point already has channels, it seems to be fairly logical that flow/current will use the existing ones rather than to make a new hole. Thus at the beginning we have homogeneous medium, after some time we get porous one.

One can see, that the system dynamics allows, in principle, formation of a cluster of any size, up to the size of the system. Obviously the larger a cluster is, the less stable it is. Let us also point out that in general case the dynamics of such systems does not have attractors, *i.e.* it can be considered as "the evolution with open end".

The proposed model seems to be close to the models investigated in the percolation theory [8,9]. Taking into account that "percolation theory deals with connectivity of a very large (macroscopic) number of elements under condition that the relation of every element with its neighbors has a random, but quite definite character, (for example, setting up by the means of using of a random number generator possess definite properties)" [9] we might emphasize that our model neither has any fixed structure of the system (that has to be understood in the sense of neighbors definition procedure) nor any determined space dimension. In this sense it is similar just to infinite-range percolation [10]. In the suggested model a structure arises in the process of system evolution, it is mobile, in this aspect it is closer rather to the second order phase transition [14]. There is the chemistry application of reversible gelation (cooking gelatine) where similar creation and destruction of bonds occurs. Let us stress that dynamics of our model essentially differs from [11,12], we do not consider a disorded lattice as well as particles motions in dynamically disorded medium. We do not study a transport task [13]. We envisage the mobile structure creation processes. But deep in our mind we are interested in the same problem – a role of connectivity in a system.

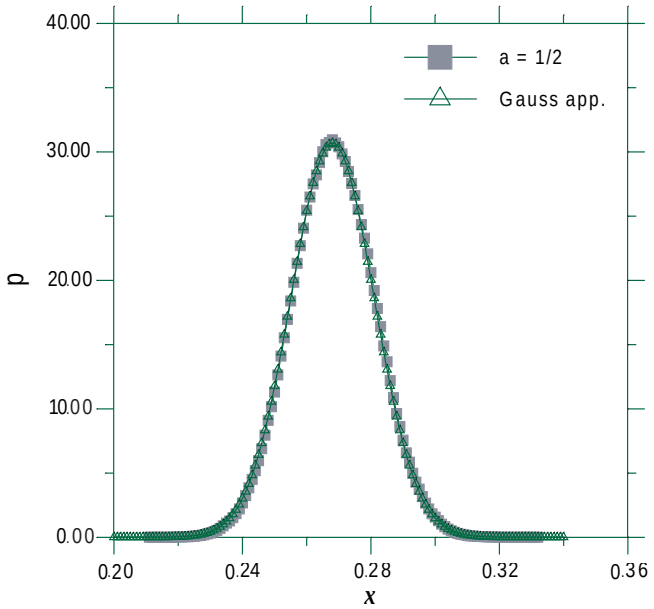

Fig. 1. Probability distribution of the system wiring p(x) for parameter $\alpha = 1/2$ (squares) in comparison with Gaussian distribution with the same parameters (triangles). Size of the system $N = 1000$.

3 Results of computer simulations

As the main characteristics of a forming structure we consider: the probability distribution of the system wiring (in this context the wiring is a ratio of the number of links to the number of elements), the average fraction of free elements (the elements which have no neighbors) in the system, the distribution of clusters as a function of size.

We have studied these characteristics for different values of parameter α. It has been turned out that the probability distribution of the system wiring is excellently approximated by the Gaussian law, *i.e.*

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} \mathrm{e}^{\frac{-(x-a)^2}{2\sigma^2}}$$

where x is the wiring (Fig. 1)
The distribution is localized on a very narrow interval of the wiring value. The results for the average and deviations for different α are given in Table 1. In Figure 2 one can see the distributions for different values of α.
Turning to the distribution of clusters as a function of its size, one can see that the character of the dependence is exponential and does not depend on a system size (Fig. 3).

$$n_s \propto \mathrm{e}^{\beta s}$$

Table 1. In this table values characterized of the properties of a forming structure for different parameter α values are presented. Where α is the probability of link appearance, *dl* - fraction of free elements in the system, *a* - average value of the wiring, σ - deviation of the wiring, β - index of the cluster size distribution, *max-s* - maximal size of a cluster, *range* - range of non-zero values of the wiring. We investigated the systems of 1000 elements ($N = 1000$).

α	dl	a	σ	β	$max-s$	$range$
1/4	0.764	0.127	0.01	−1.88235	9	[0.082;0.177]
1/3	0.689	0.172	0.0115	−1.60361	11	[0.125;0.224]
1/2	0.536	0.268	0.013	−1.24915	14	[0.211;0.332]
2/3	0.378	0.378	0.0135	−0.98363	16	[0.319;0.439]
3/4	0.295	0.442	0.013	−0.909985	18	[0.383;0.498]
0.95	0.066	0.632	0.0095	−0.725898	19	[0.590;0.669]
0.99	0.014	0.681	0.008	−0.645233	21	[0.648;0.714]

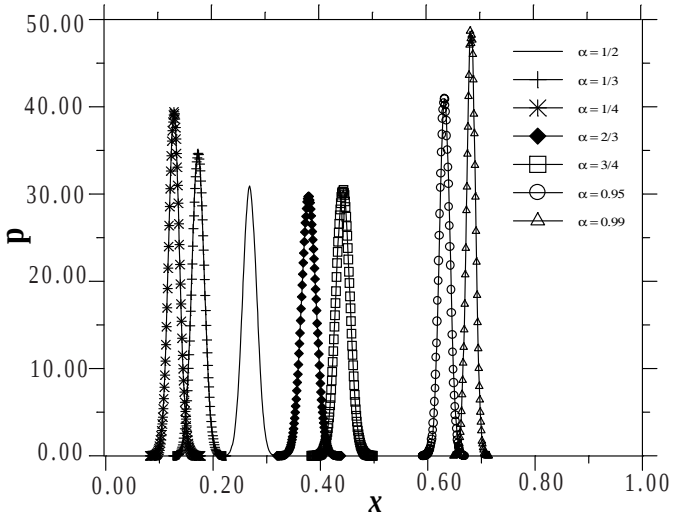

Fig. 2. Probability distribution of the system wiring $p(x)$ for parameters: $\alpha = 1/2$ (line only); $\alpha = 1/4$ (stars); $\alpha = 1/3$ (cross); $\alpha = 2/3$ (diamonds); $\alpha = 3/4$ (squares); $\alpha = 0.95$ (circles); $\alpha = 0.99$ (triangles). Size of the system $N = 1000$.

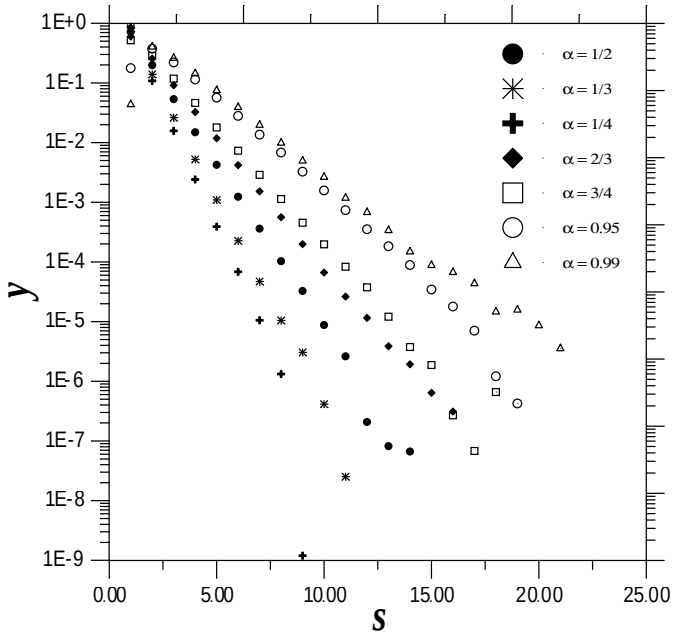

Fig. 3. Distribution of clusters size $y(s)$ for parameters: $\alpha = 1/2$ (dots); $\alpha = 1/4$ (stars); $\alpha = 1/3$ (cross); $\alpha = 2/3$ (diamonds); $\alpha = 3/4$ (squares); $\alpha = 0.95$ (circles); $\alpha = 0.99$ (triangles). Size of the system $N = 1000$.

where n_s is the number of clusters consisting of s nodes. For $1/4 \leq \alpha \leq 3/4$ the accuracy of approximation varies from 0.999067 ($\alpha = 3/4$) to 0.999862 ($\alpha = 1/2$) and for two last cases it is less: 0.998865 and 0.996851 correspondingly.

The β indices for different values α are also presented in Table 1.

We have also studied the extreme case when $\alpha = 1$. In this situation, after some transient period of time a state of the system becomes statistically stable and character of its structure does not change. In so doing, we have paid attention to the following characteristics: the relaxation time (the time of structure formation) *t-rel*, the wiring of the system w, the maximal size of a cluster *max-s*.

It was revealed that *t-rel* and *max-s* depend on the system size, but w does not. As *max-s* increases with N and taking into account that in this case the cluster distribution is better approximated by the power law with exponent equal to −2.8 (accuracy 96%) than by the exponential law with exponent equal −0.54 (accuracy 95%) we can suppose that in this quenched case our model is more similar to percolation tasks. An infinite cluster would have an infinite size. Let us note that exponent of the critical cluster size distribution in our model is greater (in absolute value) than one in the percolation models in three dimension space according to [15], where it is equal −2.2, but it has the same order.

The set of these values for different system size is presented in Table 2. The wiring value of the system for $\alpha = 1$ is equal to 0.691. Since w does not depend on the system size it should be the same in infinite systems, thus it can be envisaged as the percolation threshold in the quenched variant of our model. Comparing the obtained result with ones represented in [8] one can see that ours is fairly close to percolation threshold for honeycomb lattice, but it is rather a contingency. According to our rule of structure

Table 2. In this table the relaxation time (the time of structure construction) *t-rel* and the maximal size of a cluster *max-s* for different system size (N) in border case ($\alpha = 1$) are presented.

N	*t-rel*	*max-s*
100	446	6
200	1405	9
300	1124	11
400	2697	9
500	2502	12
600	2930	10
700	3969	8
800	4766	13
900	5700	10
1000	7012	10
1500	7675	12
2000	15367	13
2500	20632	11
3000	21785	11
3500	31820	16

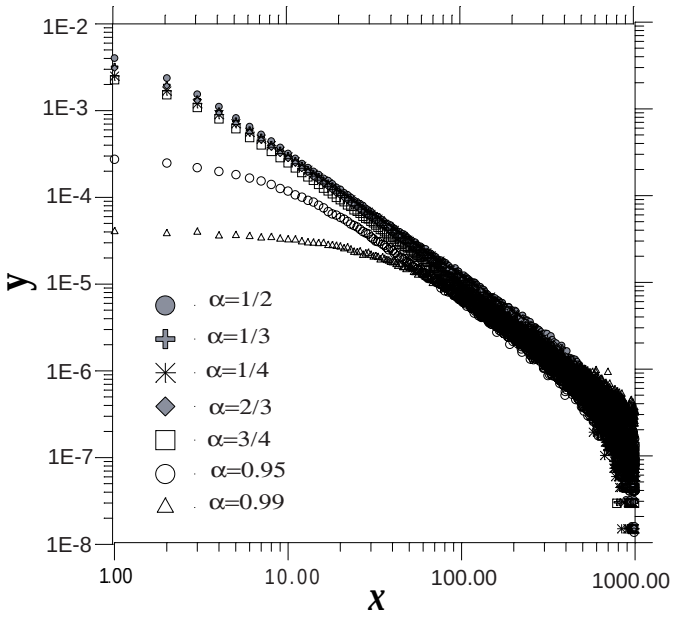

Fig. 4. Distribution of deviation from average of free elements fraction in the system $y(x)$ for parameters: $\alpha = 1/2$ (dots); $\alpha = 1/4$ (stars); $\alpha = 1/3$ (cross); $\alpha = 2/3$ (diamonds); $\alpha = 3/4$ (squares); $\alpha = 0.95$ (circles); $\alpha = 0.99$ (triangles). Size of the system $N = 500$.

Table 3. In this table one can see critical indexes γ for different values of α. $N = 500$.

α	γ
1/4	−1.42006
1/3	−1.42167
1/2	−1.42349
2/3	−1.41858
3/4	−1.42415
0.95	−1.4161
0.99	−1.42407

construction, it can be supposed a getting structure to be the variant of Bethe lattice, but in our case according to the formulae $p_c = 1/q$ (where q is number of links going from a node, p_c – percolation threshold, in our case it corresponds to w), $q \approx 1.447$, *i.e.* far away from any integer number, so one can conclude that the created structure of our system is a tree graph with no equal number of successors for every node. (According to the definition of the Bethe lattice [9] the number of successors for every node should be the same).

Because of we investigate not only the quenched variant but the possibilities both appearance and destroying of links, besides characteristics of arising structures, large interest has dynamics of the structure formation. Let us note that besides characteristics of arising structures, we can investigate dynamics of the structure formation. Namely, we are interested in such characteristics as the duration of deviation of the fraction of free elements in the system from the average.

After some transient time, the system comes to the stationary regime. Thus statistical division into two fractions, namely, the fraction of free elements and the fraction of the elements having at least a single link takes place. We count the average fraction of free elements in the system: $\lambda = \frac{1}{NT}\sum_{i=1}^{T} n_i$ where T – the total time of counting; n_i – the number of free elements at the moment i. Since each state of the system is not stable, *i.e.*, there permanently occur processes of appearing and destroying of links, it is mirrored by the change of the fraction of free elements in the system (dl) (deviation from λ). Namely, when a link appears then dl decreases, if it vanishes – dl increases. It seems to be obvious that deviations of different duration can be observed: both increase and decrease of dl. Such deviation one can think of as the avalanches in models of self-organized criticality [16,1] or as a relaxation time of the system into the quasistationary state analogously to fluctuation motions (spin fluctuations, order parameter fluctuations) in the phase transitions and the critical phenomena theory [14].

We investigated the structure destruction processes, *i.e.* the probability of the dl deviation from the average to larger values ($dl_{\rm up}$) as a function of its duration (t). In so doing, moments $t = n/N > \lambda$ and $t = n/N < \lambda$ are considered as the start and the end of an event correspondingly.

As it was shown by computer simulations, this characteristic is well approximated by the power law

$$dl_{\rm up} \sim t^{\gamma}$$

with the index $\gamma = 1.421 \pm 0.005$. In our opinion, it is a remarkable result that for $t > 3$ the curves are very close despite very different values of α: from 1/4 to 3/4. That is also true for $t > 15$ and $\alpha = 0.95$, and for $t > 50$

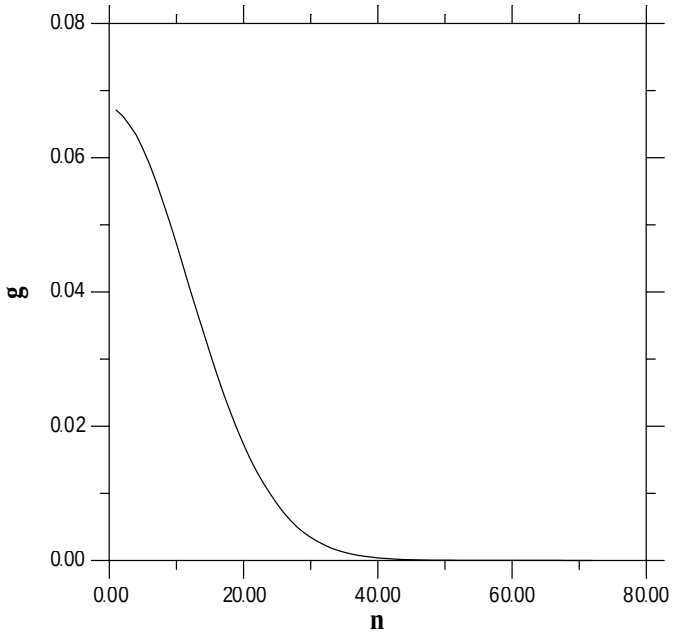

Fig. 5. Distribution of deep of deviation from average of free elements fraction in the system g(n) for $\alpha = 3/4$. Size of the system $N = 500$.

and $\alpha = 0.99$, *i.e.* the exponent (critical index) does not depend on the system parameter α. Thus we have the parameter independent critical dynamics in the system (Fig. 4). In Table 3 one can see indexes γ for different values of α.

As for cluster distributions, the best correspondence is achieved for $\alpha = 1/2$. As the deviation from this value increases, for small duration one can observe a bending of the curves to the bottom. In general, the accuracy of the approximation was near 99.7%.

We have also considered another case, *i.e.* processes of arising a structure. In this situation the results are turned out to be analogous to the former case. So the picture is symmetrical.

Apart from it we calculated distribution of deviation mass. By this term we denote deviation from the average measured in the number of elements become free or connected during an event. Analogously to the phase transitions and critical phenomena theory it can be envisage as size of fluctuation. The results one can see in (Fig. 5)

Let us stress that this result is not the only one possible. Indeed the average fraction of free elements in the system we calculated as it mentioned above as $\lambda = \frac{1}{NT}\sum_{i=1}^{T} n_i$, so it can happen that the deviation can be shorter in time and deeper in the number of elements in one direction than in another one with the same value of average.

4 Conclusion

It seems that in spite of the simplicity of its formulation, the model has a number of nontrivial characteristics.

After a transient period of time, a certain statistical mobile interaction structure is built up in the system. In the case of the quenched model ($\alpha = 1$) the distribution of cluster size is better approximated by the power law than by the exponential one, as it occurs for percolation models. Here the critical structure kind of tree graph with no equal number of successors is formed.

Due to very good correspondence of the distribution of the system wiring to the Gaussian law and taking into account that the deviation is quite small one can conclude that large fluctuations in the system occur rarely and they rapidly vanish.

The critical character of the system dynamics does not depend on introduced parameter α, as it follows from distributions of time duration of deviation from the average *dl*. The distributions are turned to be power functions with the same index $\gamma = 1.421 \pm 0.005$, so this model can be putted in the class SOC systems. It looks like the principle of structure formation employed in the model leads to a certain class of universality.

Let us stress that in the general case while system dynamics has turned out to be critical, geometry of forming structures does not have such properties, as it is shown by the exponential drop in the distribution of clusters. In addition, in this case the index depends from α.

In this work we didn't attempt to create the model of SOC system as it is done in [17], but we revealed such properties in the simplest model of structure formation. Maybe this work can be considerable as an approach to understanding of the mechanisms of self-organization to criticality.

The author greatly acknowledges K. Mardanov for the help during a preparation of the manuscript.

References

1. P. Bak, K. Sneppen, Phys. Rev. Lett. **71**, 4083 (1993); M. Paczuski, S. Maslov, P. Bak, Europhys. Lett. **27**, 97 (1994); K.A. Mardanov, Yu.M. Pismak, J. Tech. Phys. **38**, 301 (1997); Yu.M. Pismak, J. Tech. Phys. **38**, 327 (1997); O.V. Kovalev, Yu.M.Pis'mak, V.V. Vechernin, Europhys. Lett. **40**, 471 (1997).
2. K. Ho, A. Shvartsburg, B. Pan, Z. Lu, C. Wang, J. Wacker, J. Fye, M. Jarrold, Nature **392**, 582 (1998); A. Czirok, E. Ben-Jacob, I. Cohen, T. Vicsek, Phys. Rev. E **54**, 1791 (1996).
3. K. Christensen, R. Donangelo, B. Koiller, K. Sneppen, Phys. Rev. Lett. **81**, 2380 (1998).
4. P.J. Watts, S.H. Strogatz, Nature **393**, 440 (1998).
5. T. Witten, L.M. Sander, Phys. Rev. Lett. **47**, 1400 (1981); for reviews of DLA see P. Meakin, *Phase Transitions and Critical Phenomena*, Vol. 12, edited by C. Domb, J.L. Lebowitz (Academic, New York, 1988), p. 336.

6. S. Liang, S. Fuhrman, R. Somogy, *Proc. Pacific Symposium on Biocomputing 1998*, 18 (1998); J. Reinitz, D.H. Sharp, Mech. of Development **49**, 133 (1995).
7. N.C. Harary, *Structural models* (Wiley, New York, 1965).
8. A. Bunde, S. Havlin *Fractals and Disorder Systems* (Springer, Amsterdam, 1996).
9. A. Efros, *Physics and geometry of disorder* (Science, Moscow, 1982).
10. M. Gouker, F. Family, Phys. Rev. B **28**, 1449 (1983).
11. S.D. Druger, A. Nitzan, M.A. Ratner, J. Chem. Phys. **79**, 3133 (1983).
12. A.H. Harrison, R. Zvanzig, Phys. Rev. A **32**, 1072 (1985).
13. A. Bunde, M.D. Ingram, P. Maass, K.L. Ngai, J. Phys. A **24**, L881 (1991); P. Maass, A. Bunde, M.D. Ingram, Phys. Rev. Lett. **68**, 3064 (1992)
14. L.D. Landau, E.M. Lifshits *Statistical physics* (Science, Moscow 1964); S. Ma, *Modern theory of critical phenomena* (Mir, Moscow 1980).
15. C.D. Lorenz, R.M. Ziff, Phys. Rev. E **57**, 230 (1998).
16. P. Bak, C. Tang, K. Wiesenfeld, Phys. Rev. A **38**, 364 (1988).
17. H. Flyvbjerg, Phys. Rev. Lett. **76**, 940 (1996).